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A lemma for cost attained

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Abstract

A treeable ergodic equivalence relation of integer cost is generated by a free action of the free group on the corresponding number of generators. Every countable treeable ergodic equivalence relation is induced by the free action of some countable group.

Keywords: Ergodic theory; Treeable equivalence relations; Free groups; Measure equivalence of groups

1. Introduction

Given an equivalence relation E one can consider the *graphings* of E, consisting of partial functions included in the graph of E whose various compositions enable us to trace out a path between any two equivalent points. Levitt in [5] defines the cost of a measure preserving equivalence relation to be the infimum among graphings of the sums of the measures of the domains of the relevant partial functions.

Here we present a result, which in an unpublished form has been previously cited by [4] to obtain a kind of dichotomy theorem for amenability and [6] in an application to von Neumann algebras. The authors of [4] wrote up a proof of 1.1, though their organization is very different to the one below.

Proposition 1.1. *Let E be an ergodic measure preserving equivalence relation on a standard Borel probability space (X, μ); assume that every equivalence class is countable. Let Φ be a graphing for E with $C_\mu(\Phi) \geq n$.*

Then there is an alternate graphing Φ' for E which has no greater cost and contains n morphisms which are total that is to say:

(a) $C_\mu(\Phi') \leq C_\mu(\Phi)$;
(b) *and there are distinct bijections $\varphi_1, \ldots, \varphi_n$ in Φ' with $\varphi_i : X \to X$.*

One obtains additionally that if $C_\mu(\Phi) \leq n + 1$ then we may further conclude $\Phi' = \{\varphi_1, \varphi_2, \ldots, \varphi_n, \theta\}$, where $\theta : A \to B$ is a bijection, some $A, B \subset X$.

Recall that a measurable equivalence relation is *treeable* if there is a measurable way of assigning the structure of a tree to each equivalence class. In the next corollary one should bear in mind that Damien Gaboriau has shown in [3] that an E as above with finite cost is treeable if and only if it admits a graphing which actually attains its cost, and in this case any treeing will in fact realize the infimum.

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Corollary 1.2. *For E treeable and as above, the cost of E equals n if and only if there is a measure preserving action of the free group on n generators, $\mathbb{F}_n$, which is free μ-a.e. and has E as its orbit equivalence relation.*

We also show that in the case that E is treeable with infinite cost one may find a free action of $\mathbb{F}_\infty$ giving rise to E. Appealing to the connections made in [2] between orbit equivalence in the ergodic setting and measure equivalence, this implies that a countable non-amenable group is measure equivalent to a non-abelian free group if and only if it has a free measure preserving action on some standard Borel probability space which gives rise to a treeable equivalence relation.

This paper finishes with a comment on a deep theorem of Alex Furman's.

Corollary 1.3. *If E is a treeable ergodic measure preserving equivalence relation on a standard Borel probability space with countable classes, then there is a countable group G acting μ-a.e. freely giving rise to E as its orbit equivalence relation.*

Moreover, the group G can be chosen solely as a function of the cost of E.

Furman in [2] had previously obtained ergodic E which are not induced by an a.e. free action of a countable group. His examples arose by the restriction of a non-treeable equivalence relation to a non-null set, and were therefore known to be non-treeable.

2. Notation and definitions

We take all the usual notational shortcuts. All sets considered are measurable. All functions are measurable. All group actions are by measure preserving transformations. We identify functions agreeing a.e. Unless otherwise warned, the reader should assume that all non-empty sets are non-null. We tend to say *everywhere* when we only mean *almost everywhere*. $\mathbb{N}$ begins with the number 0.

Definition. A *standard Borel space* is a set X equipped with a σ-algebra $\mathcal{B}$, such that $\mathcal{B}$ is the σ-algebra generated by some choice of a Polish topology on X. A *standard Borel probability space* is a standard Borel space equipped with a probability measure on its Borel sets.

In general we will only be considering uncountable standard Borel spaces, and these are all isomorphic to the unit interval in its usual Borel structure. Thus one might reasonably think of a standard Borel probability space as just being some choice of a Borel probability measure on [0,1].

Definition. If E is an equivalence relation on a standard Borel probability space (X, μ), and $A, B \subset X$ measurable, then we say that a function

$$f : A \to B$$

is a *morphism* (for E) if it is a bijection and

$$xEf(x)$$

all $x \in A$. We say that E is *measure preserving* if every morphism is a measure preserving function. We say that E is *ergodic* if every E-invariant set is either null or conull.

From [1], the measure preserving equivalence relations with countable classes are exactly those induced by a countable group of measure preserving transformations. Even in the case that E is ergodic, [2] has shown that in general we may not be able to choose this group so that it acts *freely* on the space. Below we will prove that the additional assumption of *treeability* does ensure that we can choose the countable group so that it acts freely.

Definition. Given a set Ψ of morphisms, a *word* built from Ψ is a morphism of the form

$$x \mapsto \psi_1^{\epsilon(1)} \circ \psi_2^{\epsilon(2)} \ldots \circ \psi_n^{\epsilon(n)}(x),$$

where each $\psi_i \in \Psi$, each $\epsilon(i) \in \{-1, 1\}$, and x ranges over a set on which these compositions make sense. The word is *reduced* if at no i do we have $\epsilon(i) = -\epsilon(i+1)$ along with $\psi_i = \psi_{i+1}$. A set of morphisms Ψ is said to be a *graphing of E* if for any xEy there is a word mapping x to y; the graphing is said to be a *treeing* if there is always a *unique* reduced word. Equivalently, Φ is a treeing if the adjacency relation xRy if there exists $\varphi \in \Phi$ with $\varphi^{\pm 1}(x) = y$ providing a tree structure on each equivalence class.

For Ψ a collection of morphisms we let $\Psi^{-1} = \{\phi^{-1} : \phi \in \Psi\}$.

3. Proof

We set about proving 1.1 for $n = 2$. It should be more or less clear how to extend it to larger n. We organize this into a series of small technical lemmas, omitting proofs when they resemble earlier arguments.

The first of these lemmas, at 3.1, states that we may find a new morphism $\hat{\varphi}_0$ included in E and a corresponding partition of the space up into an infinite array of measurable sets,

$$\begin{array}{c} B_{1,0}, B_{1,1}, \\ B_{2,0}, B_{2,1}, B_{2,2}, \\ \dots \\ B_{n,0}, B_{n,1}, B_{n,2}, \dots, B_{n,n}, \\ \dots, \end{array}$$

such that at each $n \geq 1$ and $k < n$

$$\hat{\varphi}_0|_{B_{n,k}} : B_{n,k} \to B_{n,k+1}$$

is a bijection. We also want to do this in such a way as we can obtain a new graphing containing $\hat{\varphi}_0$, for which the cost has not increased, and such that all the parts of morphisms which have been lost from the older graphing can be easily recovered as powers of $\hat{\varphi}_0$.

Lemma 3.1. *Let E be as above, Φ a graphing of E. Then there is a graphing $\hat{\Phi}_0$ of E, $\hat{\varphi}_0 \in \hat{\Phi}_0$, $(B_{n,k})_{n\geq 1, k\leq n}$ a partition of X, such that:*

(1) $C_\mu(\hat{\Phi}_0) \leq C_\mu(\Phi)$ *(the cost has not increased);*
(2) $\hat{\varphi}_0[B_{n,k}] = B_{n,k+1}$ *all $k < n$ (the new morphism moves the elements of the partition in the prescribed manner);*
(3) $\mathrm{Dom}(\hat{\varphi}_0) = \bigcup_{k<n, n\in\mathbb{N}} B_{n,k}$ *(the new morphism has the indicated domain);*
(4) *for each $\varphi \in \hat{\Phi}_0$ with $\varphi \neq \hat{\varphi}_0$ we have*

$$\varphi = \psi|_C,$$

some $C \subset X$, $\psi \in \Phi$ (the new graph consists just of the new morphism and restrictions of the old morphisms);

(5) *for each $\psi \in \Phi$ there is a partition $(C_i)_{i\in\mathbb{N}}$ of X such that*
   (i) $\psi|_{C_0} \in \hat{\Phi}_0$*;*
   (ii) *and at $i > 0$, $\psi|_{C_i} = (\hat{\varphi}_0)^{\ell_i}|_{C_i}$, some $\ell_i \in \mathbb{Z}$ (we can recover the missing pieces of the old morphisms as powers of the new morphism).*

Proof. We assume that for distinct $\varphi, \varphi' \in \Phi$ we always have $\varphi(x) \neq \varphi'(x)$ when both are defined. We may also assume there is some $\hat{\psi} \in \Phi$ with $\mathrm{Ran}(\hat{\psi}) \cap \mathrm{Dom}(\hat{\psi})$ empty.

We build transfinite sequences of graphings

$$(\Psi_\alpha)_{\alpha<\delta}, (\Phi_\alpha)_{\alpha<\delta},$$

along with a choice of morphisms $\varphi_\alpha \in \Phi_\alpha$, by induction on α so that:

(a) Ψ_0 is empty; $\Phi_0 = \Phi$;
(b) Ψ_1 consists in a single morphism $\bar{\theta}_0$ with $\mathrm{Ran}(\bar{\theta}_0) \cap \mathrm{Dom}(\bar{\theta}_0) = \emptyset$, $\bar{\theta}_0 \in \Phi$, $\mu(\mathrm{Dom}(\bar{\theta}_0)) \neq 0$; we set $\varphi_0 = \bar{\theta}_0$, and for all α and $\theta \in \Psi_\alpha$ we have $\mathrm{Ran}(\theta) \cap \mathrm{Dom}(\theta)$ empty;
(c) if $\alpha + 1 < \delta$, $\alpha > 0$, and $\theta \in \Psi_{\alpha+1}$, then $\mathrm{Dom}(\theta) \subset \mathrm{Ran}(\theta')$ some $\theta' \in \Psi_\alpha$;
(d) for $\alpha \leq \beta$ we have $\Psi_\alpha \subset \Psi_\beta$ and at λ a limit ordinal we have $\Psi_\lambda = \bigcup_{\alpha<\lambda} \Psi_\alpha$;
(e) if $\varphi, \varphi' \in \Psi_\alpha$ are distinct, then their ranges are disjoint and their domain are disjoint;
(f) each $\Psi_\alpha \cup \Phi_\alpha$ graphs E;

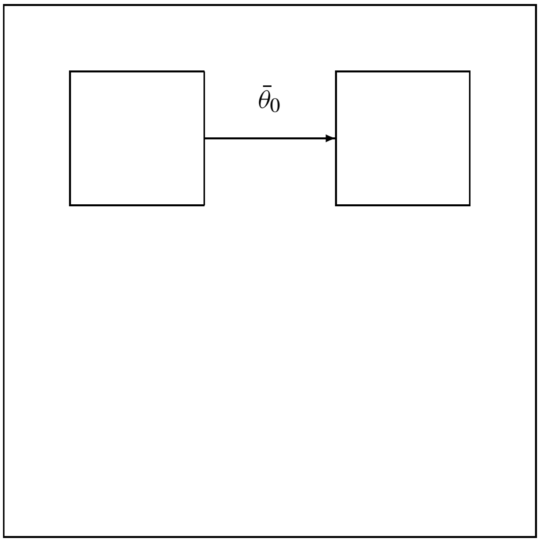


Fig. 1. We need to add again, and we find some $\phi_1 \in \Phi_1$ such that some restriction $\phi_1|_A$ or its inverse $\phi_1|_A^{-1}$ has image disjoint to both the domain and image of $\bar\theta_0$. Then, as indicated later, we can find $\hat\theta_1$ either of the form $\phi_1|_A^{\pm}$ or $\phi_1|_A^{\pm} \circ \bar\theta_1^{-1}$ whose domain is disjoint from the domain of $\hat\theta_0$ and whose range is disjoint to both its domain and range. We add $\bar\theta_1$ to Ψ_1 to obtain Ψ_2 and subtract off $\phi_1|_A$.

(g) $\varphi_\alpha \in \Phi_\alpha$ is the only morphism not appearing in $\Phi_{\alpha+1}$ and for this φ_α there is a partition of $\mathrm{Dom}(\varphi_\alpha)$ into A_0^α, A_1^α such that

$$\mu(A_1^\alpha) \neq 0;$$
$$\varphi_\alpha|_{A_1^\alpha} \in \Phi_{\alpha+1};$$

there are $\psi_1, \ldots, \psi_\ell, \hat\psi_1, \ldots, \hat\psi_k \in \Psi_\alpha \cup \Psi_\alpha^{-1}$, and $\bar\theta_\alpha \in \Psi_{\alpha+1}$, with

$$\mathrm{Dom}(\bar\theta_\alpha) = \hat\psi_1 \circ \ldots \hat\psi_k[A_0^\alpha]$$

and we have either

$$\varphi_\alpha|_{A_0^\alpha} = \psi_1 \circ \psi_2 \circ \ldots \psi_\ell \circ \bar\theta_\alpha \circ \hat\psi_1 \circ \ldots \circ \hat\psi_k|_{A_0^\alpha}$$

or

$$(\varphi_\alpha|_{A_0^\alpha})^{-1} = \psi_1 \circ \psi_2 \circ \ldots \psi_\ell \circ \bar\theta_\alpha \circ \hat\psi_1 \circ \ldots \circ \hat\psi_k|_{A_0^\alpha}$$

(h) if λ is a limit then for $\alpha < \lambda$ and $\varphi \in \Phi_\alpha$, we have $\varphi \in \Phi_\lambda$ if φ is in every earlier Φ_α, or otherwise we have $\varphi|_{A^\lambda_{\infty,\varphi}} \in \Phi_\lambda$, where

$$A^\lambda_{\infty,\varphi} = \bigcap\{A_1^\beta : \beta \geq \alpha, \varphi_\beta \subset \varphi\}.$$

Inrough terms, we begin the construction by taking some $\bar\theta_0$ in our original graphing which can be assumed to have disjoint range and image and simply adding it to Ψ_1 and subtracting it from Φ_0 to obtain Φ_1. We just describe the first few steps, without giving much in the way of proofs yet.

Thus the general idea of this construction is to steadily transfer across pieces of the Φ_α's to the Ψ_α's, so that $\Phi_\alpha \cup \Psi_\alpha$ continues to graph E. The crucial part of this is (g). It tells us that when we remove a single piece $\varphi_\alpha|_{A_0^\alpha}$ of a morphism $\varphi_\alpha \in \Phi_\alpha$ then we are compensating by placing into $\Psi_{\alpha+1}$ a morphism, $\bar\theta_\alpha$, which can reconstruct $\varphi_\alpha|_{A_0^\alpha}$ using only *pre-existing morphisms* already placed into Ψ_α. As we continue through the construction, and survey the construction at ever larger ordinals β, the Ψ_β sets only get bigger, and our ability to write $\varphi_\alpha|_{A_0^\alpha}$ as a word from Ψ_β is never endangered.

The other parts of this construction are less vital. (a) and (d) are bookkeeping requirements, describing how we add to the Ψ_α's and remove from the Φ_α's. (f) actually follows from the other clauses. (e) ensures that partial morphisms in

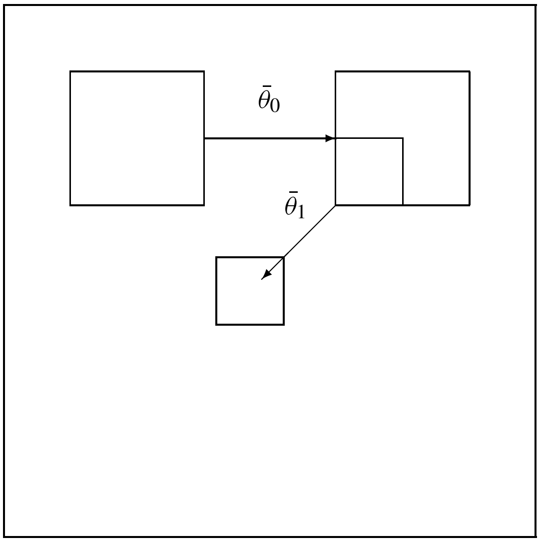


Fig. 2. We insist that $\bar{\theta}_0, \bar{\theta}_1$ have disjoint domains and that the range of $\bar{\theta}_1$ is entirely new. After this we keep going, finding some $\bar{\theta}_2$ whose domain is disjoint from the domains of $\bar{\theta}_0$ and $\bar{\theta}_1$ and is, relative to these two morphisms, interdefinable with some element of Φ_2. We do insist that it picks up some more of the space in its image, though we do not object if its domain is included in the *range* of one of the earlier functions.

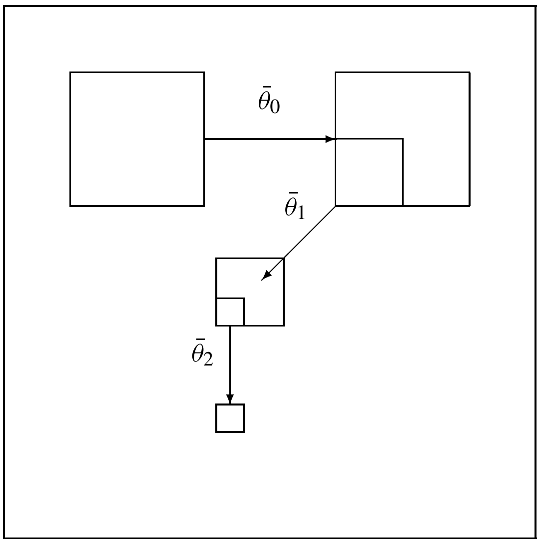


Fig. 3. The domain of $\bar{\theta}_2$ is included in the image of the earlier functions, but its image is new. After this we keep going to add $\bar{\theta}_3$, spreading out to new domains and so on.

the Ψ_α's will eventually have some morphism as their union, which in turn will give us $\hat{\varphi}_0$ described in the statement of 3.1. (b) and (c) will enable us to obtain the $(B_{n,k})$ sets with the structure indicated above. (h) and (d) state that at limit ordinals we take an appropriate limit of the process so far, with a union on the Ψ_α side and a kind of "intersection" along the Φ_α's.

We continue with this construction for as long as possible, eventually arriving at some $(\Psi_\alpha)_{\alpha<\delta}$, $(\Phi_\alpha)_{\alpha<\delta}$ admitting no further extension. We will argue that this final ordinal δ is a successor ordinal, $\delta = \beta + 1$, and that in some natural way $\Psi_\beta \cup \Phi_\beta$ will yield $\hat{\varphi}_0$ and $\hat{\Phi}_0$ as required.

Claim (1). *δ is not a limit ordinal.*

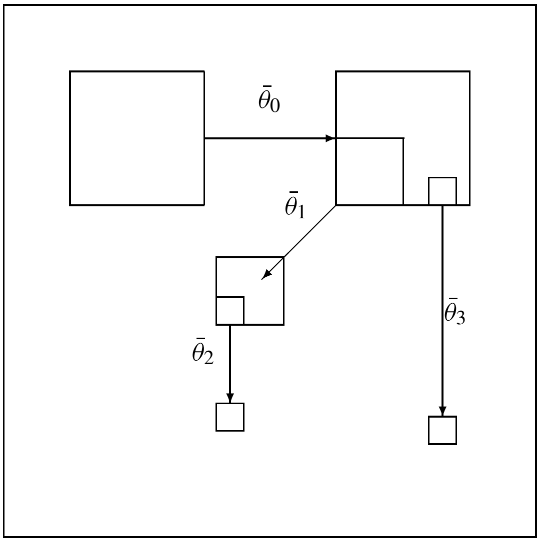


Fig. 4. The next morphism $\bar{\theta}_3$ takes its domain from the image of $\bar{\theta}_0$. We do not rule out returning to the images of much earlier morphisms.

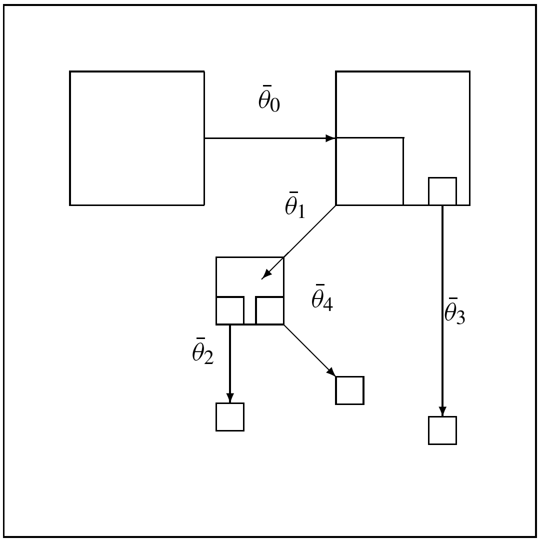


Fig. 5. In this way we eventually ensure that all the space except for $\mathrm{Dom}(\bar{\theta}_0)$ is in the range of some $\bar{\theta}_\alpha$.

Proof of Claim. Otherwise we could simply let $\Psi_\delta = \bigcup_{\alpha<\delta} \Psi_\alpha$, and let Φ_δ consist of all $\varphi|_{A^\delta_{\infty,\varphi}}$ where $\varphi \in \Phi_\alpha$ some $\alpha < \delta$ and as in (h) above $A^\delta_{\infty,\varphi} = \bigcap\{A^\beta_1 : \beta \geq \alpha, \beta < \delta, \varphi_\beta \subset \varphi\}$. (□Claim)

So from now on let us fix α with $\alpha + 1 = \delta$. For each $\beta \leq \alpha$ let $B^\beta_0 = \mathrm{Dom}(\bar{\theta}_0)$, and for each $n \in \mathbb{N}$ let $B^\beta_{n+1} = \bigcup\{\theta[B^\beta_n] | \theta \in \Psi_\beta\}$.

Claim (2). *For $\beta \leq \alpha$ and $n \neq m$ we have B^β_n, B^β_m disjoint.*

Proof of Claim. By clause (e) in our construction and transfinite induction on β. (□Claim)

Claim (3). *For $\beta \leq \alpha$ and $\theta \in \Psi_\beta$ we have* $\mathrm{Dom}(\theta) \subset \bigcup_{n\in\mathbb{N}} B_n^\beta$*; and thus*

$$\bigcup_{n\in\mathbb{N}} B_n^\beta = \bigcup_{\theta\in\Psi_\beta} \mathrm{Dom}(\theta) \cup \mathrm{Ran}(\theta).$$

Proof of Claim. By clause (c) in our construction and induction on β. (□Claim)

Claim (4). *For $\beta \leq \alpha$ and a.e. $x \in \bigcup_{n\in\mathbb{N}} B_n^\beta$ either:*

(1) $x \notin \mathrm{Dom}(\theta)$ *all $\theta \in \Psi_\beta$; or*
(2) *there exists k and $\theta_0, \theta_1, \ldots, \theta_k \in \Psi_\beta$ such that $\theta_k \circ \theta_{k-1} \circ \ldots \circ \theta_0(x)$ is well defined and not a member of* $\mathrm{Dom}(\theta)$ *any $\theta \in \Psi_\beta$.*

Proof of Claim. Otherwise suppose not; we define $B_{n,\infty}^\beta$ to be the set of $x \in B_n^\beta$ such that for every k there exists $\theta_0, \theta_1, \ldots \theta_k \in \Psi_\beta$ with $\theta_k \circ \theta_{k-1} \circ \ldots \circ \theta_0(x)$ well defined, and observe that this set will have positive measure. It then follows by clause (c) of our construction that we may at each m define a morphism from $B_{m,\infty}^\beta$ to $B_{m+1,\infty}^\beta$ and thus for $m \leq m'$ we have $\mu(B_{m,\infty}^\beta) \leq \mu(B_{m',\infty}^\beta)$, and thus $(B_{m,\infty}^\beta)_{m\geq n}$ provides a sequence of disjoint sets with measure bounded away from zero, and a contradiction to $\mu(X) = 1$. (□Claim)

The next claim uses ergodicity for the first time.

Claim (5). *X equals the union of the* $\mathrm{Dom}(\theta)$, $\mathrm{Ran}(\theta)$ *for $\theta \in \Psi_\alpha$.*

Proof of Claim. Otherwise by ergodicity of E and Claims (3) and (4) we may find some $\varphi \in \Phi_\alpha^{\pm 1}$, $m \in \mathbb{N}$, and non-null A such that

$$\varphi[A] \cap \bigcup_{n\in\mathbb{N}} B_n^\alpha = \emptyset,$$
$$A \subset \mathrm{Dom}(\varphi) \cap B_m^\alpha$$

and either

(1) $\mathrm{Dom}(\theta) \cap A = \emptyset$ all $\theta \in \Psi_\alpha$; or
(2) there exists k and $\theta_0, \theta_1, \ldots, \theta_k \in \Psi_\alpha$ such that $\theta_k \circ \theta_{k-1} \circ \ldots \circ \theta_0(x)$ is well defined all $x \in A$ and $\theta_k \circ \theta_{k-1} \circ \ldots \circ \theta_0[A]$ and $\mathrm{Dom}(\theta)$ are disjoint for any $\theta \in \Psi_\alpha$.
We assume that (2) holds and that $\varphi \in \Phi_\alpha$; the other cases are exactly similar.

We can then let $\bar{\theta}_\alpha$ have domain $\theta_k \circ \theta_{k-1} \circ \ldots \circ \theta_0[A]$ and set

$$\bar{\theta}_\alpha(x) = \varphi \circ \theta_0^{-1} \circ \theta_1^{-1} \circ \ldots \theta_k^{-1}(x).$$

We let $A_0^\alpha = A$, $A_1^\alpha = \mathrm{Dom}(\varphi) \setminus A_0^\alpha$, $\Psi_{\alpha+1} = \Psi_\alpha \cup \{\bar{\theta}_\alpha\}$, $\Phi_{\alpha+1} = (\Phi_\alpha \setminus \{\varphi\}) \cup \{\varphi|_{A_1^\alpha}\}$. In this way we are able to contest another round, contradicting the assumption that the construction ground to a halt at δ. (□Claim)

We can then finish up the proof of the lemma by letting $B_{n,k}$ be the set of $x \in B_k^\alpha$ such that there are $\theta_1, \theta_2, \ldots, \theta_{n-k} \in \Psi_\alpha$ with $\theta_1 \circ \ldots \circ \theta_{n-k}(x)$ well defined and n is the largest such integer. (In other words, $\bigcup_{k\leq n} B_{n,k}$ is the set of elements whose orbit under Ψ_α has size exactly $n + 1$.) We use the disjointness of the morphisms in Ψ_α to define the longed for $\hat{\varphi}_0$: for $x \in B_{n,k}$ we consider the unique $\theta \in \Psi_\alpha$ with $x \in \mathrm{Dom}(\theta)$ and let $\hat{\varphi}_0(x) = \theta(x)$. □

We now let $B = \bigcup_{n\in\mathbb{N}} B_{n,0}$. We are going to repeat the previous step, relativizing the process to B.

Lemma 3.2. *There is a graphing $\hat{\Phi}^*$ of E, containing $\hat{\varphi}_0$ along with a new morphism $\hat{\varphi}^*$, with $(C_{n,k})_{n\in\mathbb{N},k\leq n}$ a partition of B, such that:*

(1) $C_\mu(\hat{\Phi}^*) \leq C_\mu(\hat{\Phi}_0)$;
(2) $\hat{\varphi}^*[C_{n,k}] = C_{n,k+1}$ *all $k < n$;*
(3) $\mathrm{Dom}(\hat{\varphi}^*) = \bigcup_{k<n,n\in\mathbb{N}} C_{n,k}$*;*

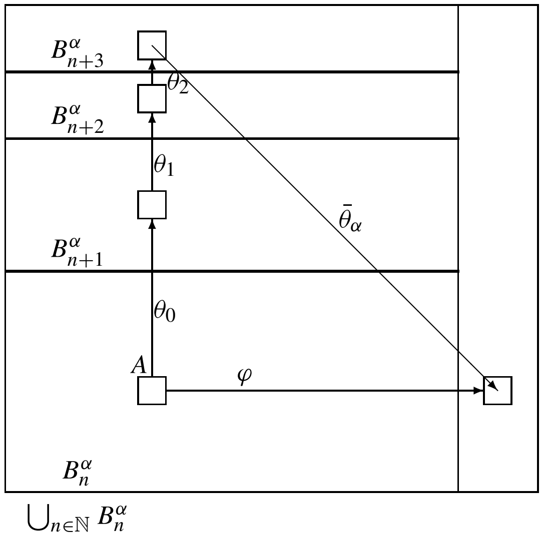


Fig. 6. A typical case: the image of φ is what we want, but the domain intersects the domain of morphisms already in Ψ_α.

(4) *for each* $\varphi \in \hat{\Phi}^*$ *with* $\varphi \neq \hat{\varphi}^*, \hat{\varphi}_0$ *we have*

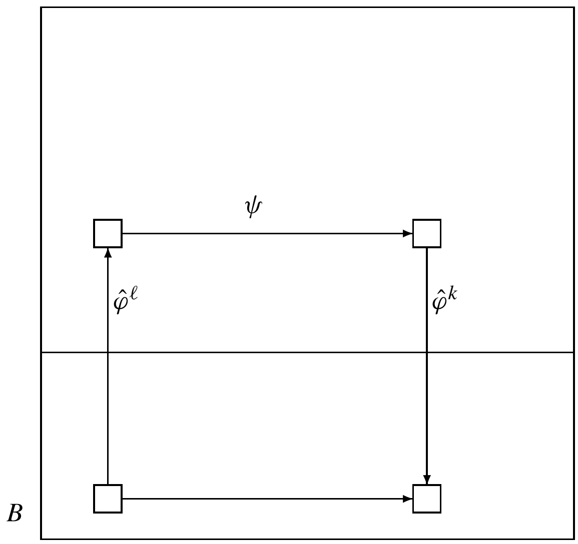
$$\varphi = \psi|_C,$$

some $C \subset X$, $\psi \in \hat{\Phi}_0$;

(5) *for each* $\psi \in \hat{\Phi}_0$ *there is a partition* $(D_i)_{i\in\mathbb{N}}$ *of* X *such that*
 (i) $\psi|_{D_0} \in \hat{\Phi}^*$;
 (ii) *and at* $i > 0$, $\psi|_{C_i}$ *equals some word built up from* $\hat{\varphi}^*, \hat{\varphi}_0$ *restricted to* C_i.

Proof. We may first of all assume without any loss of generality that for each $\psi \in \hat{\Phi}_0 \setminus \{\hat{\varphi}_0\}$ there are k, ℓ with

$$\mathrm{Dom}((\hat{\varphi}_0)^k \circ \psi \circ (\hat{\varphi}_0)^\ell), \mathrm{Ran}((\hat{\varphi}_0)^k \circ \psi \circ (\hat{\varphi}_0)^\ell) \subset B.$$



We may then consider the graphing $\hat{\Phi}^\sharp$ which for each $\psi \in \hat{\Phi}_0$, $\psi \neq \hat{\varphi}_0$, has the appropriate morphism $(\hat{\varphi}_0)^k \circ \psi \circ (\hat{\varphi}_0)^\ell$ for $E|_B$.

With this granted it is straightforward to check that 3.1 applied to $E|_B$, $\mu_B = \frac{\mu|_B}{\mu(B)}$, $\hat{\Phi}^\sharp$, produces the requisite $\hat{\varphi}^*$. $\square$

Now we can define a new morphism $\hat{\varphi}_1$ with domain

$$\bigcup_{n\in\mathbb{N},k<n} B_{n,k} \cup \left\{ x | \exists n \left(x \in B_{n,n}, \hat{\varphi}_0^{-n}(x) \in \bigcup_{m\in\mathbb{N},k<m} C_{m,k} \right) \right\}.$$

This morphism $\hat{\varphi}_1$ will extend the old $\hat{\varphi}_0$, and so we simply set $\hat{\varphi}_1(x) = \hat{\varphi}_0(x)$ for $x \in \bigcup_{n\in\mathbb{N},k<n} B_{n,k}$. For $x \in B_{n,n}$ and $\hat{\varphi}_0^{-n}(x) \in \bigcup_{m\in\mathbb{N},k<m} C_{m,k}$ we let

$$\hat{\varphi}_1(x) = \hat{\varphi}^* \circ \hat{\varphi}_0^{-n}(x).$$

From this we obtain a new graphing $\hat{\Phi}_1$ of E with

$$\hat{\Phi}_1 = (\hat{\Phi}^* \setminus \{\hat{\varphi}_0, \hat{\varphi}^*\}) \cup \{\hat{\varphi}_1\}.$$

Comparing $\hat{\Phi}_1$ with $\hat{\Phi}_0$ and $\hat{\varphi}_1$ with $\hat{\varphi}_0$ we discover the following:

$C_\mu(\hat{\Phi}_1) \le C_\mu(\hat{\Phi}_0)$;
$\hat{\Phi}_1$ is still a graphing of E;
$\hat{\varphi}_1$ extends $\hat{\varphi}_0$ and $\mu(X \setminus \mathrm{Dom}(\hat{\varphi}_1)) \le \frac{1}{2}\mu(X \setminus \mathrm{Dom}(\hat{\varphi}_0))$;
for any $\psi \in \hat{\Phi}_0$ we can partition $\mathrm{Dom}(\psi)$ into $(D_i)_{i\in\mathbb{N}}$ such that
(a) $\psi|_{D_0} \in \hat{\Phi}_1$;
(b) for each $i > 0$ there is $\ell_i \in \mathbb{Z}$ such that $\psi|_{D_i} = (\hat{\varphi}_1)^{\ell_i}|_{D_i}$.

Plainly we can continue this indefinitely, obtaining a sequence $(\hat{\Phi}_n, \hat{\varphi}_n)_{n\in\mathbb{N}}$ where at each n

$C_\mu(\hat{\Phi}_n) \le C_\mu(\hat{\Phi})$;
$\hat{\Phi}_n$ is a graphing of E;
$\hat{\varphi}_n$ extends $\hat{\varphi}_{n-1}$ and $\mu(\mathrm{Dom}(\hat{\varphi}_n)) \ge 1 - 2^{-n}$;
for any $\psi \in \hat{\Phi}_{n-1}$ we can partition $\mathrm{Dom}(\psi)$ into $(D_i)_{i\in\mathbb{N}}$ such that
(a) $\psi|_{D_0} \in \hat{\Phi}_n$;
(b) for each $i > 0$ there is $\ell_i \in \mathbb{Z}$ such that $\psi|_{D_i} = (\hat{\varphi}_n)^{\ell_\infty}|_{D_i}$.

In the end we let

$$\hat{\varphi}_\infty = \bigcup_{n\in\mathbb{N}} \hat{\varphi}_n$$

and for each $\psi \in \hat{\Phi}_1$ we place into $\hat{\Phi}_\infty$ the morphism

$$\psi|_{A_{\infty,\psi}},$$

where $A_{\infty,\psi}$ equals

$$\bigcap\{D : \exists n \exists \psi' \in \hat{\Phi}_n(\psi' \subset \psi, D = \mathrm{Dom}(\psi'))\}.$$

By considering the measure of the domain we actually have

$$\hat{\varphi}_\infty : X \to X$$

(almost everywhere defined). By the nature of the definition of $\hat{\Phi}_\infty$ and the assumptions on the various $\hat{\Phi}_n$ we have that for any $\psi \in \hat{\Phi}_n$ we can partition $\mathrm{Dom}(\psi)$ into $(D_i)_{i\in\mathbb{N}}$ such that

(a) $\psi|_{D_0} \in \hat{\Phi}_\infty$;
(b) for each $i > 0$ there is $\ell_i \in \mathbb{Z}$ such that $\psi|_{D_i} = (\hat{\varphi}_1)^{\ell_i}|_{D_i}$.

This gives us a new graphing $\hat{\Phi}_\infty$ of E containing a morphism $\hat{\varphi}_\infty : X \to X$ with $C_\mu(\hat{\Phi}_\infty) \le C_\mu(\Phi)$. We are now going to take that whole step over again, adding in a new morphism but keeping hold of $\hat{\varphi}_\infty$ and not allowing it to be changed. This requires relativizing 3.1 to $\hat{\varphi}_\infty$.

Lemma 3.3. *Let E, $\hat{\Phi}_\infty$, $\hat{\varphi}_\infty$, be as above. Then there is a graphing $\hat{\Theta}_0$ of E which still contains $\hat{\varphi}_\infty$, and a morphism $\hat{\theta}_0 \in \hat{\Theta}_0$, and $(D_{n,k})_{n\in\mathbb{N},k\le n}$ a partition of X, such that:*

(1) $C_\mu(\hat{\Theta}_0) \le C_\mu(\hat{\Phi}_\infty)$;
(2) $\hat{\theta}_0[D_{n,k}] = D_{n,k+1}$ *all* $k < n$;
(3) $\mathrm{Dom}(\hat{\theta}_0) = \bigcup_{k<n,n\in\mathbb{N}} D_{n,k}$;
(4) *for each* $\varphi \in \hat{\Theta}_0$ *with* $\varphi \neq \hat{\varphi}_\infty, \hat{\theta}_0$ *we have*

$$\varphi = \psi|_C,$$

some $C \subset X$, $\psi \in \hat{\Phi}_\infty$;
(5) *for each* $\psi \in \hat{\Phi}_\infty$ *there is a partition* $(C_i)_{i\in\mathbb{N}}$ *of* X *such that*
  (i) $\psi|_{C_0} \in \hat{\Theta}_0$;
  (ii) *and at* $i > 0$, $\psi|_{C_i}$ *equals some word in* $\hat{\theta}_0, \hat{\varphi}_\infty$ *restricted to* C_i.

Proof. This closely parallels the proof of 3.1. There is a difference in how we show we can continue at inductive steps.

We build graphings

$$(\Theta_\alpha)_{\alpha<\delta}, (\Gamma_\alpha)_{\alpha<\delta},$$

and morphisms $\theta_\alpha \in \Gamma_\alpha$:

(a) Θ_0 is empty; $\Gamma_0 = \hat{\Phi}_\infty \setminus \{\hat{\varphi}_\infty\}$;
(b) Θ_1 consists in a single morphism $\hat{\rho}$, where for some k, ℓ, and $A = \hat{\varphi}_\infty^{-k}[\mathrm{Dom}(\rho)]$, we have

$$\hat{\varphi}_\infty^{\ell} \circ \hat{\rho} \circ \hat{\varphi}_\infty^{k}|_A \in \hat{\Phi}_\infty;$$

for all $\alpha < \delta$ and $\theta \in \Theta_\alpha$ we have $\mathrm{Ran}(\theta) \cap \mathrm{Dom}(\theta)$ empty;
(c) if $\alpha + 1 < \delta$, $\alpha > 0$, and $\theta \in \Theta_{\alpha+1}$, then $\mathrm{Dom}(\theta) \subset \mathrm{Ran}(\theta')$ some $\theta' \in \Theta_\alpha$;
(d) for $\alpha \le \beta$ we have $\Theta_\alpha \subset \Theta_\beta$ and at λ a limit ordinal we have $\Theta_\lambda = \bigcup_{\alpha<\lambda} \Theta_\alpha$;
(e) if $\varphi, \varphi' \in \Theta_\alpha$ are distinct, then their ranges are disjoint and their domains are disjoint;
(f) each $\Theta_\alpha \cup \Gamma_\alpha \cup \{\hat{\varphi}_\infty\}$ graphs E;
(g) $\theta_\alpha \in \Gamma_\alpha$ is the only morphism not appearing in $\Gamma_{\alpha+1}$ and for this θ_α there is a partition of $\mathrm{Dom}(\theta_\alpha)$ into A_0^α, A_1^α such that

$$\theta_\alpha|_{A_1^\alpha} \in \Gamma_{\alpha+1};$$

there are $\psi_1, \ldots, \psi_\ell, \hat{\psi}_1, \ldots, \hat{\psi}_k \in (\Theta_\alpha \cup \{\hat{\varphi}_\infty\})^{\pm}$, and $\bar{\theta}_\alpha \in \Theta_{\alpha+1}$, with

$$\mathrm{Dom}(\bar{\theta}_\alpha) = \hat{\psi}_1 \circ \ldots \hat{\psi}_k[A_0^\alpha]$$

and we have either

$$\theta_\alpha|_{A_0^\alpha} = \psi_1 \circ \psi_2 \circ \ldots \psi_\ell \circ \bar{\theta}_\alpha \circ \hat{\psi}_1 \circ \ldots \circ \hat{\psi}_k|_{A_0^\alpha}$$

or

$$(\theta_\alpha|_{A_0^\alpha})^{-1} = \psi_1 \circ \psi_2 \circ \ldots \psi_\ell \circ \bar{\theta}_\alpha \circ \hat{\psi}_1 \circ \ldots \circ \hat{\psi}_k|_{A_0^\alpha}$$

(h) if λ is a limit then for $\alpha < \lambda$ and $\varphi \in \Gamma_\alpha$, we either have $\varphi \in \Gamma_\lambda$ or we have $\varphi|_{A^\lambda_{\infty,\varphi}} \in \Gamma_\lambda$, where

$$A^\lambda_{\infty,\varphi} = \bigcap \{A_1^\beta : \beta \ge \alpha, \theta_\beta \subset \varphi\}.$$

Again this construction stops at some successor ordinal $\delta = \alpha + 1$, and as before at each $\beta \leq \alpha$ and $n \in \mathbb{N}$ we can let

$$D_0^\beta = \mathrm{Dom}(\hat{\rho}),$$
$$D_{n+1}^\beta = \{\theta[D_n^\beta] : \theta \in \Theta_\beta, n \in \mathbb{N}\}.$$

Again $n \neq m$ implies D_n^β and D_m^β are disjoint. And again $\bigcup\{\mathrm{Dom}(\theta) \cup \mathrm{Ran}(\theta) : \theta \in \Theta_\beta\}$ equals $\bigcup_{n\in\mathbb{N}} D_n^\beta$. And again the real battle is to show that $\bigcup_{n\in\mathbb{N}} D_n^\alpha = X$, and for this time around we have some more work. Suppose $\bigcup_{n\in\mathbb{N}} D_n^\alpha \neq X$, and we try to show that after all we could have continued to define $\Theta_{\alpha+1}, \Gamma_{\alpha+1}, \theta_\alpha, \bar{\theta}_\alpha$.

Definition. Let F be the equivalence relation on X induced by the graphing $\Theta_\alpha \cup \{\hat{\varphi}_\infty\}$.

Case 1. F is ergodic.

Then we can choose some $\theta_\alpha \in \Gamma_\alpha$, words $\psi, \hat{\psi}$ built up from $\Theta_\alpha \cup \{\hat{\varphi}_\infty\}$, $A_0^\alpha \cup A_1^\alpha$ partitioning $\mathrm{Dom}(\theta_\alpha)$, with

$$\hat{\psi}^{-1}[A_0^\alpha] \subset \bigcup_{n\in\mathbb{N}} D_n^\alpha \setminus \bigcup\{\mathrm{Dom}(\theta) | \theta \in \Theta_\alpha\}$$
$$\psi \circ \theta_\alpha[A_0^\alpha] \subset X \setminus \bigcup_{n\in\mathbb{N}} D_n^\alpha.$$

After shrinking we may assume $A_0^\alpha \subset \mathrm{Ran}(\theta)$ some single $\theta \in \Theta_\alpha$, and then we can let $\bar{\theta}_\alpha = \psi \circ \theta_\alpha \circ \hat{\psi}|_{\hat{\psi}^{-1}[A_0^\alpha]}$.

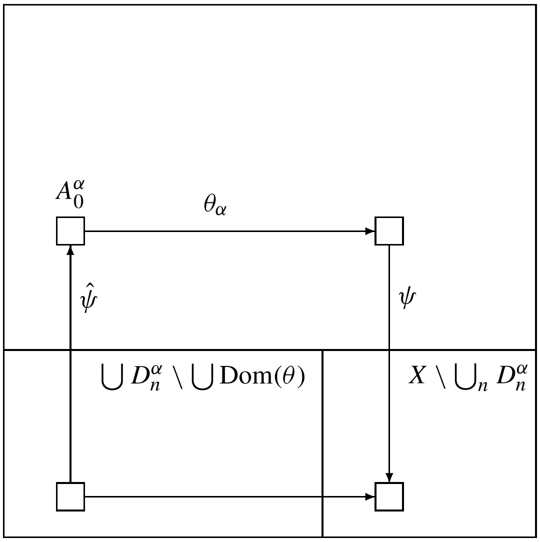


Fig. 7.

Case 2. F is not ergodic.

Then it follows that we may find $Y_1 \subset \bigcup_{n\in\mathbb{N}} D_n^\alpha$, $Y_2 \subset X \setminus (\bigcup_{n\in\mathbb{N}} D_n^\alpha)$,

$$0 < \mu(Y_1), \mu(Y_2),$$

and for all $y \in Y_1$ the equivalence class $[y]_F$ is disjoint from Y_2.

However E is ergodic and graphed by $\Theta_\alpha \cup \Gamma_\alpha \cup \{\hat{\varphi}_\infty\}$, and so we may find words $\psi_1, \psi_2, \ldots, \psi_\ell$ from $\Theta_\alpha \cup \{\hat{\varphi}_\infty\}$ and $\tau_1, \tau_2, \ldots, \tau_{\ell-1} \in \Gamma_\alpha$ with

$$\psi_\ell \circ \tau_{\ell-1} \circ \psi_{\ell-1} \circ \tau_{\ell-2} \ldots \circ \psi_1[Y_1] \subset X \setminus \bigcup_{n\in\mathbb{N}} D_n^\alpha,$$

$$\psi_{\ell-1} \circ \tau_{\ell-2} \ldots \circ \psi_1[Y_1] \subset \bigcup_{n\in\mathbb{N}} D_n^\alpha.$$

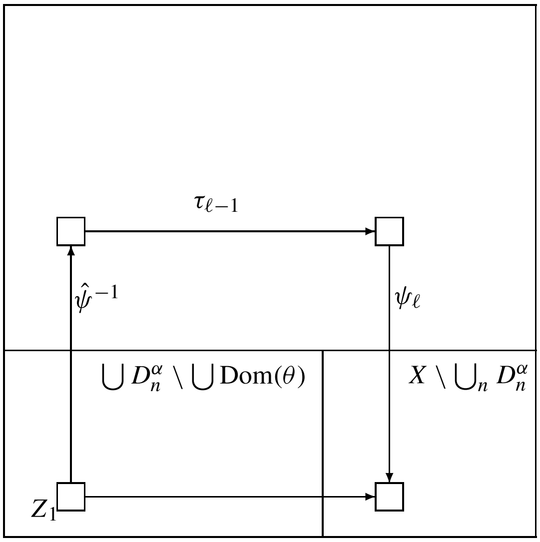


Fig. 8.

Again after possibly refining Y_1 we may assume there is some word $\hat{\psi}$ from Θ_α such that

$$\hat{\psi} \circ \psi_{\ell-1} \circ \tau_{\ell-2} \circ \psi_{\ell-2} \ldots \circ \psi_1[Y_1] \subset \bigcup_{n\in\mathbb{N}} D^\alpha_n \setminus \bigcup\{\mathrm{Dom}(\theta) | \theta \in \Theta_\alpha\}.$$

And we go onto another round with

$$\bar{\theta}_\alpha = \psi_\ell \circ \tau_{\ell-1} \circ \hat{\psi}^{-1}|_{Z_1},$$
$$\theta_\alpha = \tau_{\ell-1},$$

where $Z_1 = \hat{\psi} \circ \tau_{\ell-2} \circ \ldots \psi_1[Y_1]$. $\square$

We then let $D = \bigcup_{n\in\mathbb{N}} D_{n,0}$.

Here is probably a good point to pause and formulate the general result. The proof of this general technical lemma clearly follows from the above argument.

Lemma 3.4. *Let F be an ergodic measure preserving equivalence relation on standard Borel space (Y, ν), with all classes countable, ν a finite Borel measure. Let Θ be a graphing of F containing morphisms $\{\psi_1, \psi_2, \ldots, \psi_n\}$. Let $A \subset Y$ be a set whose saturation under $\{\psi_1, \psi_2, \ldots, \psi_n\}$ is conull — that is to say, for almost all $x \in Y$ there is some $y \in A$ and word φ_w from $\{\psi_1, \psi_2, \ldots, \psi_n\}$ with $\varphi_w(y) = x$. Suppose further more that $C_\nu(\Theta) \geq C_\nu(\{\psi_1, \psi_2, \ldots, \psi_n\}) + \mu(A)$ and there is some $\bar{\psi} \in \Theta \setminus \{\psi_1, \psi_2, \ldots, \psi_n\}$ having* $\mathrm{Dom}(\bar{\psi})$, $\mathrm{Ran}(\bar{\psi})$ *disjoint subsets of A.*

Then there is a graphing Θ^* of F and a morphism ψ^* for F such that:

(1) *$C_\nu(\Theta^*) \leq C_\nu(\Theta)$; and if Θ is a treeing then so too is Θ^*;*
(2) *for some partition $(Y_{n,k})_{n\in\mathbb{N},k\leq n}$ of A we have*

$$\psi^*[Y_{n,k}] = Y_{n,k+1}$$

all $k < n$, $n \in \mathbb{N}$;
(3) $\mathrm{Dom}(\psi^*) = \bigcup_{n\in\mathbb{N},k<n} Y_{n,k}$*; and $\psi^* \supset \bar{\psi}$;*
(4) *for each $\varphi \in \Theta^* \setminus \{\psi_1, \ldots, \psi_n, \psi^*\}$ we have $\varphi = \psi|_C$ some $C \subset Y$, $\psi \in \Theta$;*
(5) *for $\psi \in \Theta$ there is a partition $(C_i)_{i\in\mathbb{N}}$ such that*
  (i) *$\psi|_{C_0} \in \Theta^*$;*
  (ii) *at $i > 0$, $\psi|_{C_i}$ equals some word in $\{\psi_1, \psi_2, \ldots, \psi_n, \psi^*\}$ restricted to C_i.*

With this general formulation observed and set to one side, let us continue with the proof for the specific case in front of us.

Lemma 3.5. *There is a graphing $\hat{\Theta}_0^*$ of E, containing $\hat{\varphi}_\infty$, $\hat{\theta}_0$, along with a new morphism $\hat{\theta}^*$, and there is $(H_{n,k})_{n\in\mathbb{N},k\leq n}$, a partition of D, such that:*

(1) $C_\mu(\hat{\Theta}_0^*) \leq C_\mu(\hat{\Theta}_0)$;
(2) $\hat{\theta}^*[H_{n,k}] = H_{n,k+1}$ *all* $k < n$;
(3) $\mathrm{Dom}(\hat{\theta}^*) = \bigcup_{k<n,n\in\mathbb{N}} H_{n,k}$;
(4) *for each* $\varphi \in \hat{\Theta}^*$ *with* $\varphi \neq \hat{\theta}^*, \hat{\theta}_0, \hat{\varphi}_\infty$ *we have*

$$\varphi = \psi|_C,$$

some $C \subset X$, $\psi \in \hat{\Theta}_0$;
(5) *for each* $\psi \in \hat{\Theta}_0$ *there is a partition* $(D_i)_{i\in\mathbb{N}}$ *of* X *such that*
 (i) $\psi|_{D_0} \in \hat{\Theta}_0$;
 (ii) *and at* $i > 0$, $\psi|_{C_i}$ *equals some word built up from* $\hat{\theta}^*, \hat{\theta}_0, \hat{\varphi}_\infty$ *restricted to* C_i.

Proof. We apply the lemma to $\hat{\Theta}_0$ for Θ, D for A, $\{\hat{\varphi}_\infty, \hat{\theta}_0\}$ for $\{\psi_1, \ldots, \psi_n\}$ to obtain $(H_{n,k})_{n\in\mathbb{N},k\leq n}$ partitioning A and $\hat{\theta}^*$ as required. $\square$

With this claim granted, we can mimic earlier arguments and choose $\hat{\theta}_0^* \supset \hat{\theta}^*$ with $\mu(\mathrm{Dom}(\hat{\theta}_0^*)) = \mu(\mathrm{Dom}(\hat{\theta}^*)) + \mu(\mathrm{Dom}(\hat{\theta}^*))$, and $\{\hat{\theta}^*\}$ graphing the same equivalence relation as $\{\hat{\theta}^*, \hat{\theta}_0\}$. And then we may clearly continue with this over and over, obtaining at each n $\hat{\Theta}_n^*$ and $\hat{\theta}_n^*$ such that:

$C_\mu(\hat{\Theta}_n^*) \leq C_\mu(\hat{\Theta}_0)$;
$\hat{\Theta}_n^*$ is a graphing of E containing $\hat{\theta}_n^*, \hat{\varphi}_\infty$;
$\hat{\theta}_n^*$ extends $\hat{\theta}_{n-1}^*$ and $\mu(\mathrm{Dom}(\hat{\theta}_n^*)) \geq 1 - 2^{-n}$;
for any $\psi \in \hat{\Theta}_0$ we can partition $\mathrm{Dom}(\psi)$ into $(D_i)_{i\in\mathbb{N}}$ such that
(a) $\psi|_{D_0} \in \hat{\Theta}_n^*$;
(b) for each $i > 0$ $\psi|_{D_i}$ can be written in a word in $\hat{\theta}_n^*, \hat{\varphi}_\infty$.

Continuing in this fashion we obtain some

$$\hat{\theta}_\infty = \bigcup_n \hat{\theta}_n^*,$$

and as before we may define Φ' to be the appropriate limit of the $\hat{\Theta}_n^*$, thereby completing the proof of 1.1 in the case that $C_\mu(\Phi) \geq 2$.

The general case of cost greater than some arbitrary integer is clearly exactly similar. We may also observe that the last step from this argument suggests the following modification:

Lemma 3.6. *Let F be an ergodic measure preserving equivalence relation on standard Borel space (Y, ν), with all classes countable, ν a finite Borel measure. Let Θ be a graphing of F containing morphisms $\{\psi_1, \psi_2, \ldots, \psi_n\}$. Let $A \subset Y$ be a set whose saturation under $\{\psi_1, \psi_2, \ldots, \psi_n\}$ is conull — that is to say, for almost all $x \in Y$ there is some $y \in A$ and word φ_w from $\{\psi_1, \psi_2, \ldots, \psi_n\}$ with $\varphi_w(y) = x$. Suppose furthermore that $C_\nu(\Theta) \geq C_\nu(\{\psi_1, \psi_2, \ldots, \psi_n\}) + \mu(A)$ and there is some $\bar{\psi} \in \Theta \setminus \{\psi_1, \psi_2, \ldots, \psi_n\}$ having* $\mathrm{Dom}(\bar{\psi})$, $\mathrm{Ran}(\bar{\psi})$ *disjoint subsets of A.*

Then there is a graphing Θ^* of F and a morphism ψ^* for F such that:

(1) $C_\nu(\Theta^*) \leq C_\nu(\Theta)$; *and if* Θ *is a treeing then so too is* Θ^*;
(2) $\psi^* : A \to A$; $\psi^* \supset \bar{\psi}$;
(3) *for each* $\varphi \in \Theta^* \setminus \{\psi_1, \ldots, \psi_n, \psi^*\}$ *we have* $\varphi = \psi|_C$ *some* $C \subset Y$, $\psi \in \Theta$;
(4) *for* $\psi \in \Theta$ *there is a partition* $(C_i)_{i\in\mathbb{N}}$ *such that*
 (i) $\psi|_{C_0} \in \Theta^*$;
 (ii) *at* $i > 0$, $\psi|_{C_i}$ *equals some word in* $\{\psi_1, \psi_2, \ldots, \psi_n, \psi^*\}$ *restricted to* C_i.

4. Corollaries

Lemma 4.1. *A treeable ergodic measure preserving equivalence relation E with countable classes on a standard Borel probability space has cost n if and only if it is induced by some free action of $\mathbb{F}_n$.*

Proof. The *if* direction is known from [3], so we concentrate on the converse.

We begin with a treeing Φ of E; by [3], $C_\mu(\Phi) = n$. Applying the argument of the last section we can find an alternative graphing Θ containing $\theta_1, \ldots, \theta_n$, each

$$\theta_i : X \to X,$$

and with

$$C_\mu(\Theta) \leq C_\mu(\Phi).$$

Since $n = C_\mu(E) \leq C_\mu(\Theta) \leq C_\mu(\Phi) = n$, we have equality throughout and hence $\Theta = \{\theta_1, \ldots, \theta_n\}$. □

Lemma 4.2. *Let E be an ergodic measure preserving equivalence relation with countable classes on a standard Borel probability space (X, μ). If E has infinite cost and is treeable, then there is a free action of $\mathbb{F}_\infty$ giving rise to E as its orbit equivalence relation.*

Proof. Let $\Theta = \{\varphi_1, \ldots, \varphi_n, \ldots\}$ be a treeing of E with infinite cost. Without loss we may assume that each $\mathrm{Dom}(\varphi_n)$ is disjoint from $\mathrm{Ran}(\varphi_n)$.

Then iterating Lemma 3.6 we may find successive treeings

$$\Theta_1, \Theta_2, \ldots, \Theta_m, \ldots$$

and morphism $\psi_1, \ldots, \psi_m, \ldots$ and measurable sets $(A_{n,m})_{m<n\in\mathbb{N}}$ such that

(a) each $\Theta_m = \{\psi_1, \ldots \psi_m, \varphi_{m+1}|_{A_{m+1,m}}, \varphi_{m+2}|_{A_{m+2,m}}, \ldots\}$;
(b) each $\psi_i : X \to X$ is total;
(c) each φ_m can be written as a word in $\{\psi_1, \ldots, \psi_m\}$;
(d) for $n > m$ we may partition $\mathrm{Dom}(\varphi_m)$ up into $(B_i)_{i\in\mathbb{N}}$ such that
  (i) $B_0 = A_{n,m}$, and so $\varphi_n|_{B_0} \in \Theta_m$;
  (ii) each $\varphi_n|_{B_i}$ can be written as the restriction of a word in $\{\psi_1, \ldots, \psi_m\}$.

We finish with $\{\psi_i : i \in \mathbb{N}\}$ as a graphing of E. Since each Θ_m is a treeing so too is the limit, $\{\psi_i : i \in \mathbb{N}\}$. Since each ψ_i is total and since they jointly give rise to a treeing, we thus obtain the free action of $\mathbb{F}_\infty$. □

Lemma 4.3. *Let E be an ergodic measure preserving equivalence relation with countable classes on a standard Borel probability space (X, μ). If E is treeable, then there is a free action of a countable group G giving rise to E as its orbit equivalence relation.*

Proof. We at once can assume the cost is finite, or else the result follows with $G = \mathbb{F}_\infty$ from the last lemma. By earlier results we may assume there is a treeing Θ and some $\varphi \in \Theta$ which is total. By Dye's theorem on the orbit equivalence of ergodic $\mathbb{Z}$-actions, we can assume that there is a sequence of subsets of X, $(A_i)_{i\in\mathbb{N}}$, such that each

$$\mu(A_i) = 2^{-i},$$
$$A_{i+1} \subset A_i,$$
$$\varphi^{2^i} : A_{i+1} \to A_i,$$

and $\{A_{i+1}, \varphi^{2^i}[A_{i+1}]\}$ partitioning A_i. At each i we let $\varphi_i : A_{i+1} \to A_i$ be given by

$$\varphi_i = \varphi^{2^i}|_{A_{i+1}};$$

note that $\{\varphi_i : i \in \mathbb{N}\}$ graphs the same equivalence relation as $\{\varphi\}$.

We then build $(k_i)_{i\in\mathbb{N}}$, $k_0 \in \mathbb{N}$, each $k_{i+1} \in \{0, 1\}$, and morphisms

$$\psi_{i,j} : A_i \to A_i$$

for $j < k_i$, and treeings Θ_n for E such that:

(a) $\Theta_n = \{\varphi_i : i \in \mathbb{N}\} \cup \{\psi_{i,j} : i \le n, j < k_i\} \cup \{\theta|_{B_{\theta,n}} : \theta \in \Theta\}$, each $B_{\theta,n}$ some subset of $\mathrm{Dom}(\theta)$;
(b) $C_\mu(\{\varphi_i : i \in \mathbb{N}\} \cup \{\psi_{i,j} : i \le n, j < k_i\}) \ge C_\mu(\Theta) - 2^{-n-1} = C_\mu(E) - 2^{-n-1}$;
(c) for each $\theta \in \Theta$ we may partition X into $(C_i)_{i\in\mathbb{N}}$ such that
   (i) $\theta|_{C_0} \in \Theta_n$;
   (ii) each $\theta|_{C_{i+1}}$ equals some word in $\{\varphi_i : i \in \mathbb{N}\} \cup \{\psi_{i,j} : i \le n, j < k_i\}$.

It follows from Lemma 3.6 that we may indeed construct such a sequence. Given the graphing $\hat{\Theta}_n = \{\varphi_i : i \in \mathbb{N}\} \cup \{\psi_{i,j} : i \le n, j < k_i\}$, or the morally equivalent graphing $\bar{\Theta}_n = \{\varphi\} \cup \{\psi_{i,j} : i \le n, j < k_i\}$, we can consider whether $C_\mu(E) \ge C_\mu(\hat{\Theta}_n) + 2^{-n-1}$. If no, we just pass on with $k_{n+1} = 0$; if yes, then we apply 3.6 to obtain some

$$\psi_{n+1,0} : A_{n+1} \to A_{n+1}$$

and graphing

$$\Theta_{n+1} = \{\varphi_i : i \in \mathbb{N}\} \cup \{\psi_{i,j} : i \le n+1, j < k_i\} \cup \{\theta|_{B_{\theta,n+1}} : \theta \in \Theta\},$$

and take the process to the next round.

This construction granted it follows from (c) that

$$\Theta_\infty = \{\varphi_i : i \in \mathbb{N}\} \cup \{\psi_{i,j} : i \in \mathbb{N}, j < k_i\}$$

graphs E. Since each Θ_n is a treeing it follows that Θ_∞ is a treeing. We will use it define a group action in some natural way.

We let G be the group with generators $\{a_i : i \in \mathbb{N}\}$, $\{b_{i,j} : i \in \mathbb{N}, j < k_i\}$. We will ask that this group be free subject to the relations

$$a_\ell b_{i,j} = b_{i,j} a_\ell$$

for $i > \ell$,

$$(a_\ell)^2 = 1,$$
$$a_\ell a_k = a_k a_\ell$$

all k, ℓ. For each $i \in \mathbb{N}$ we define a total function $T_i : X \to X$ by first choosing for a.e. x the unique $m_0^x, m_1^x, \ldots, m_{i-1}^x \in \{-1, 0\}$ such that

$$y_x = \varphi_{i-1}^{m_{i-1}^x} \circ \varphi_{i-2}^{m_{i-2}^x} \circ \ldots \varphi_0^{m_0^x}(x) \in A_i$$

and then letting

$$T_i(x) = \varphi_0^{-m_0^x} \circ \varphi_1^{-m_1^x} \circ \ldots \varphi_{i-1}^{-m_{i-1}^x} \circ \varphi_i(y_x)$$

if $y_x \in A_{i+1}$,

$$T_i(x) = \varphi_0^{-m_0^x} \circ \varphi_1^{-m_1^x} \circ \ldots \varphi_{i-1}^{-m_{i-1}^x} \circ \varphi_i^{-1}(y_x)$$

if $y_x \in A_i \setminus A_{i+1}$. (In other words we recursively define each T_i to be the unique $T_i : X \to X$ of order 2 which extends φ_i and commutes with T_j all $j < i$.) Similarly we define for $j < k_i$

$$S_{i,j}(x) = \varphi_0^{-m_0^x} \circ \varphi_1^{-m_1^x} \circ \ldots \varphi_{i-1}^{-m_{i-1}^x} \circ \psi_{i,j}(y_x).$$

We want to show that if we let each a_ℓ act on X via T_ℓ and each $b_{i,j}$ act via $S_{i,j}$ then firstly it is well defined as an action of G and secondly that it is free a.e.

Claim (1).

$$S_{i,j} \circ T_\ell = T_\ell \circ S_{i,j}$$

all $\ell < i$,

$$T_\ell = T_\ell^{-1}$$
$$T_\ell \circ T_k = T_k \circ T_\ell$$

all ℓ, k.

Proof of Claim. This follows quickly from the definitions. (□Claim)

Thus this assignment $a_\ell \mapsto T_\ell$, $b_{i,j} \mapsto S_{i,j}$ extends to a homomorphism

$$G \to M_\infty(X),$$
$$g \mapsto \varphi_g,$$

where $M_\infty(X)$ is the group of invertible mpts on X and gives us a measure preserving action of G on X.

Claim (2). *If $\varphi_g(x) = x$ for a non-null collection of $x \in X$ then $g = 1$.*

Proof of Claim. Suppose g is as above and for a non-null set of x we have $\varphi_g(x) = x$. We attempt to reduce g down to 1 using the relations imposed as part of the definition of G.

We may write the group element in the form

$$g = c_0 c_1 \ldots c_M$$

where each c_k equals either $a_{i(k)}$, $b_{i(k),j(k)}$, or $b^{-1}_{i(k),j(k)}$. After possibly replacing each c_k by a suitable

$$a_0^{-m_0} a_1^{-m_1} \ldots a_{i(k)-1}^{-m_{i(k)-1}} c_k a_{i(k)-1}^{m_{i(k)-1}} \ldots a_0^{m_0}$$

we may assume that there is a positive measure set $A \subset X$ such that for all $x \in A$

$$\varphi_g(x) = x$$

and at each $k \leq M$ we have

$$U_{k+1} \circ U_{k+2} \circ \ldots \circ U_M(x) \in A_{k(i)},$$

where each U_ℓ is respectively $T_{i(\ell)}$, $S_{i(\ell),j(\ell)}$, $S^{-1}_{i(\ell),j(\ell)}$, depending on whether c_ℓ equals either $a_{i(\ell)}$, $b_{i(\ell),j(\ell)}$, or $b^{-1}_{i(\ell),j(\ell)}$; the key point is that in G the element c_k equals $a_0^{-m_0} a_1^{-m_1} \ldots a_{i(k)-1}^{-m_{i(k)-1}} c_k a_{i(k)-1}^{m_{i(k)-1}} \ldots a_0^{m_0}$ as a consequence of the relations imposed in the definition of the group G. It then follows from Θ_∞ being a treeing that we may reduce the word

$$U_0 \circ U_1 \circ \ldots U_M|_A$$

down to the identity by the operations of canceling various $U_\ell \circ U_{\ell+1}$ when $U_{\ell+1} = U_\ell^{-1}$. Thus in particular it follows that $c_0 c_1 \ldots c_M$ will easily reduce to the identity in G and we are done. (□Claim) □

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